Problems

Tarea 2

5.1 $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ are matrices generated on the same set of basis functions by the operators A, B, C. Show that

(a) if $C = A + B$, then $\tilde{C} = \tilde{A} + \tilde{B}$,

(b) if $C = AB$, then $\tilde{C} = \tilde{A}\tilde{B}$.

5.2 (a) Using the functions ϕ_1, ϕ_0, ϕ_{-1} in Problem 4.2 as basis, calculate the matrices $\tilde{L}_x$, $\tilde{L}_y$, $\tilde{L}_z$, $\tilde{L}^2$.

(b) Show that the eigenvalues of $\tilde{L}_x$ and $\tilde{L}^2$ have the expected values.

5.3 The Pauli spin operators σ_x, σ_y, σ_z are defined in terms of the spin angular momentum operators S_x, S_y, S_z by $\sigma_x = 2S_x/\hbar$, and similarly for y and z. Consider the case $s = \frac{1}{2}$, and denote the normalised eigenfunctions of S_z by α and β. Use the relations for the raising and lowering operators (5.33) and (5.34) to prove the following results.

(a) $\sigma_x\alpha = \beta, \quad \sigma_y\alpha = \mathrm{i}\beta, \quad \sigma_z\alpha = \alpha,$

$\sigma_x\beta = \alpha, \quad \sigma_y\beta = -\mathrm{i}\alpha, \quad \sigma_z\beta = -\beta.$

(b) The normalised eigenfunctions of σ_x, σ_y, σ_z are

operator	eigenfunctions	
σ_x	$(\alpha + \beta)/\sqrt{2}$	$(\alpha - \beta)/\sqrt{2}$
σ_y	$(\alpha + \mathrm{i}\beta)/\sqrt{2}$	$(\alpha - \mathrm{i}\beta)/\sqrt{2}$
σ_z	α	β

The eigenvalues are +1 for the eigenfunctions in the left-hand column, and −1 for those in the right-hand column.

(c) The matrices of σ_x, σ_y, σ_z on the basis of α and β are

$$\tilde{\sigma}_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \tilde{\sigma}_y = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \quad \tilde{\sigma}_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

5.4 P is a beam of atoms with spin quantum number $\frac{1}{2}$ and zero orbital angular momentum, all with angular momentum $+\hbar/2$ along the x axis. Q is a beam of similar but unpolarised atoms.

(a) What is the spin state function of P in terms of α and β, the eigenfunctions of S_z?

(b) If the two beams are passed separately through a Stern–Gerlach apparatus with its magnetic field along the z axis, is there any difference between the emerging beams in the two cases?

(c) How could the difference between P and Q be detected experimentally?

5.5 The beam Q in the last problem is an incoherent mixture of the states α and β in equal proportions. Its spin state function may therefore be written as

$$\psi = (e_1\alpha + e_2\beta)\sqrt{2},$$

where e_1 and e_2 are to be regarded as complex numbers of modulus unity with random relative phases, i.e. they satisfy the relations

$$|e_1|^2 = |e_2|^2 = 1, \quad \langle e_1^* e_2 \rangle = \langle e_2^* e_1 \rangle = 0,$$

where the brackets $\langle\ \rangle$ indicate the average over all values of the relative phase. By expressing ψ in terms of the eigenfunctions of S_x, show that this state function gives the required physical result, namely, that if an unpolarised beam of the atoms is passed through a Stern–Gerlach apparatus with its magnetic field in the x direction, the two emerging beams contain equal numbers of atoms.

5.6 A beam of atoms with spin quantum number $\frac{1}{2}$ and zero orbital angular momentum passes through a Stern–Gerlach magnet whose magnetic field is along a direction D at an angle θ to the z axis. The emerging beam with spins along D is passed through a second Stern–Gerlach magnet with its magnetic field along the z axis. Show that in the two beams that emerge from the second magnet the numbers of atoms with spins parallel and anti-parallel to the z axis are in the ratio $\cos^2\frac{1}{2}\theta : \sin^2\frac{1}{2}\theta$.

5.7 (a) Let **S** be the operator for the resultant spin angular momentum of two electrons, and S_z its z component. If Φ_{SM} is an eigenfunction of S^2 and S_z with respective eigenvalues $S(S+1)\hbar^2$ and $M\hbar$, derive the expression for each Φ_{SM} in terms of the product functions $\alpha\alpha$, $\alpha\beta$, $\beta\alpha$, $\beta\beta$, where the first α or β refers to electron 1, and the second to electron 2.

(b) Consider the addition of an orbital angular momentum **L** and a spin angular momentum **S** for the case $l = 1$, $s = \frac{1}{2}$. The eigenfunctions of the operators L^2, S^2, L_z, S_z are products of the space functions ϕ_1, ϕ_0, ϕ_{-1} and the spin functions α, β. Derive the eigenfunctions Φ_{jm_j} of the operators L^2, S^2, J^2, J_z in terms of the first set of eigenfunctions.
[Hint: In both cases start with the Φ function in which S, M (or j, m_j) have their maximum values, and apply the appropriate lowering operators to both sides of the equation.]